Nonlinear stability of vortex pairs

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Figure : counterrotating vortex pair in real life

Euler equations

Euler Equations for incompressible, non-viscous (ideal) fluid flow:

$$\begin{cases} \partial_t u + (u \cdot \nabla) u = -\nabla p, \\ \operatorname{div} u = 0, \\ \lim_{|x| \to \infty} u(\cdot, x) = 0, \\ u(t = 0) = u^0, \end{cases}$$

wher u is the velocity and p is the scalar pressure of the flow.
We introduce the vorticity $\omega =$ curl u. In general, the velocity u can be expressed in terms of ω by the Biot-Savart law, which in operator form we write as $u = K[\omega]$.

Vorticity formulation

The vorticity formulation of the Euler equations is obtained by taking the curl of the velocity equations. In 2D we get:

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0, \\ u = K[\omega]. \end{cases}$$

Biot-Savart law: If the fluid domain is $\mathbb{R}^2$ then $K[\omega] = K * \omega$ with

$$K(x) = \frac{x^\perp}{2\pi |x|^2} = \frac{(-x_2, x_1)}{2\pi |x|^2}.$$

Known results in $\mathbb{R}^2$

- McGrath 1968: u_0 smooth ($H^s(\mathbb{R}^2)$, $s > 2$) then $\exists$, !, continuous dependence.
- Yudovich 1963: $\omega_0 \in L^\infty$ in a bounded domain then $\exists$ and !. Extended to include $L^1 \cap L^\infty(\mathbb{R}^2)$ by Majda (1982). Proofs yield continuous dependence as well. Extended to slightly unbounded vorticities by Yudovich (1995) and Vishik (1999).
- Existence of weak solutions for initial vorticities in $\mathcal{BM}_c^+ \cap H^{-1}_{loc}$ by Delort (1990) and for certain reflection symmetric changing-sign data by L., Nussenzveig Lopes and Xin (2001). Nonuniqueness in the context of wild solutions by Dellelis and Szekelyhidi (2011).

Examples

- Point vortices: $\omega = \sum_j m_j \delta(x - P_j)$. Hence,

$$u = u(t,x) = \sum_{\ell} m_\ell K(x - P_\ell) = \sum_{\ell} m_\ell \frac{(x - P_\ell)^\perp}{|x - P_\ell|^2}.$$

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- Vortex patches: $\omega = m\chi_D$, $D = D(t)$. Hence,

$$u = u(t,x) = \int_{D(t)} mK(x - y)\, dy.$$

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- Vortex sheets: $\omega_0 = \gamma_0 \delta_{C_0}$, where C_0 is a curve in the plane.

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Steady x stationary - stationary means time-independent solution; steady means stationary modulo the action of the group of rigid motions.

Orbital stability

Orbital stability: appropriate notion of stability for non-dissipative systems. For each initial state ω_0 let $\omega = \omega(t)$ be the evolution of our system with initial data ω_0.

Orbital stabilty of stationary states: Fix ω_* a stationary state. It is orbitally stable if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\mathrm{dist}(\omega_0, \omega_*) < \delta$ implies $\sup_{t>0} \mathrm{dist}(\omega(t) - \omega_*) < \epsilon$.

Orbital stability of steady states: More complicated - fix $\omega_* = \omega_*(t)$ a steady state. Denote $\mathcal{O} = \{\omega_*(t), t \in \mathbb{R}\}$ the *orbit* of ω_*. The state ω_* is orbitally stable if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\mathrm{dist}(\omega_0, \mathcal{O}) < \delta$ implies $\sup_{t>0} \mathrm{dist}(\omega(t) - \mathcal{O}) < \epsilon$.

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Domain of KAM theory

History - XIX century view

Rayleigh's criterion - flows of the form $(u(x_2), 0)$ on an infinite straight horizontal channel (which are always stationary) are *linearly stable* if and only if the velocity profile has no inflection point.

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Basic tool - finding eigenvalues of the linearization

History - Arnold's Theorem

Theorem

(Arnold 65) Let Ω be a smooth bounded domain in the plane and let u_* be a C^3 stationary solution of the Euler equations in Ω. Suppose that there exists $c_1, c_2 > 0$ such that one of the two conditions hold:

1. $$c_1 \leq -\frac{u_*}{\nabla^\perp \omega_*} \leq c_2$$

2. $$c_1 \leq \frac{u_*}{\nabla^\perp \omega_*} \leq c_2,$$

   with c_1 big enough (wrt lowest eingenvalue of Laplacian),

then u_* is stable with respect to the norm $\|u\|_{L^2} + \|\omega\|_{L^2}$.

Basic tool - find a potential well

History - extending Arnold's result

Basic avenues:

- Relax regularity condition on the stationary solution on bounded domains, ultimately to include vortex patches (Burton 2005); approach based on Kelvin's variational principle - extrema of the kinetic energy within rearrangement classes of vorticity are stable.

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- Relax boundedness of the domain - go to full space (Wan and Pulvirenti 85) nonlinear stability in $\|\omega\|_{L^1}$ for circular vortex patches and monotone decreasing circularly symmetric vorticity configurations in the plane, using the moment of inertia

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  a conserved quantity.
- Go for steady, instead of stationary vorticity configurations - full space, adapting Burton's techniques with Kelvin's variational principle - subject of this talk, Burton, L^2, 2013.

History - Counterrotating vortex pairs

CVP - solutions of 2D Euler, vorticity odd with respect to the x_2 variable, propagating with constant speed along the x_1-axis.

- first studied by Pocklington, 1895.
- Existence proved by Norbury (1975), Yang (1991) as solutions to nonlinear eigenvalue problem (smooth case) and by Burton (1988) by minimization of kinetic energy within rearrangement classes of vorticity - vast family of examples, at least one in each rearrangement class.
- Numerical (Overman and Zabuski, 1982) and experimental (Duc and Sommeria, 1988) work on vortex pairs.
- Stability of vortex pairs observed in experiments, formally studied by Pierrehumbert (1980) conjectured by Saffman (1995).

Rearrangement classes

Definition

Let $\Pi = \{x_2 > 0\}$, the upper half-plane. For $\omega \in L^1(\Pi)$ we define the distribution function $\lambda_\omega = \lambda_\omega(s) = |\{x \in \Pi : |\omega(x)| > s\}|$. We say that two functions ω_1 and ω_2 are *rearrangements* if $\lambda_{\omega_1} = \lambda_{\omega_2}$ a.e.. We denote by $\mathcal{R}(\omega)$ the set of all functions in $L^1(\Pi)$ which are rearrangements of ω. We denote by $\overline{\mathcal{R}}(\omega)$ the closure of $\mathcal{R}(\omega)$ with respect to the weak topology of $L^2(\Pi)$.

Remark 1: Euler evolution preserves the rearrangement class.
Remark 2: The set $\overline{\mathcal{R}}(\omega)$ is much larger than $\mathcal{R}(\omega)$, in particular, it is convex (Douglas, 94).

Variational principle

For $\omega \in L^1_c(\Pi)$ we define $\widetilde{\omega}$ the odd extension of ω to the full plane, and $u = K * \widetilde{\omega}$ the flow velocity associated with $\widetilde{\omega}$. We define

$$E[\omega] \equiv \int_\Pi |u|^2 dx \text{ and } I[\omega] \equiv \int_\Pi x_2 \omega dx.$$

We will look for stable vortex pairs among maximizers of the functional $E - \lambda I$ on rearrangement classes. This is an adaptation of Kelvin's variational principle to this context.

Main Theorem

We introduce the norm $\|\omega\|_Y = \|\omega\|_{L^2} + |I[\omega]|$.

Theorem

(Burton, L2,2013) Let ω_0 be a nonnegative function in $L^p(\Pi)_c$, for $2 < p \leq \infty$ and fix $\lambda > 0$. Let Σ_λ denote the set of maximizers of $E - \lambda I$ on $\overline{\mathcal{R}}(\omega_0)$. Assume that $\emptyset \neq \Sigma_\lambda \subseteq \mathcal{R}(\omega_0)$. Then Σ_λ is orbitally stable in the following sense. For every $\epsilon > 0$ and $A > |\mathrm{supp}(\omega_0)|$ there exists a $\delta > 0$ such that any Euler trajectory $\omega = \omega(t)$ with $\omega(0)$ nonnegative, compactly supported, $|\mathrm{supp}(\omega(0))| < A$ and the Y-distance between $\omega(0)$ and Σ_λ is less than δ then the L^2 distance between $\omega(t)$ and Σ_λ is less than ϵ forever.

Observations

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3. Σ_λ is a compact set of functions in L^2, together with their translates.
4. For the special case of Lamb's circular vortex the maximizer of $E - \lambda I$ includes the function 0, which means this case is not covered by our result.

the proof, 1

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Compactness, plus some confinement analysis shows that the approximating sequence converges strongly in L^2 to someone in Σ_λ, which basically is enough to conclude.

the proof, 2

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For a nonnegative function $\omega \in L^1(\Pi)$, we define ω^* as the Steiner symmetrization for each fixed x_2. Clearly, ω^* is a rearrangement of ω and $I[\omega] = I[\omega^*]$. In addition, a symmetrization inequality holds: $E(\omega^*) \geq E(\omega)$.

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These facts, applied to the approximation sequence ω^n show no dychotomy and explains why maximizers are Steiner symmetric.

Comments and conclusions

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- THANK YOU!