Optimal minimax bounds for the Navier-Stokes equations

Ricardo M. S. Rosa

Instituto de Matemática
Universidade Federal do Rio de Janeiro

Seminário de Análise e EDP - IM - UFRJ

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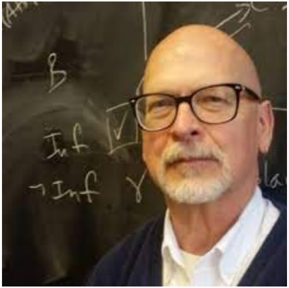
Work

- **“Optimal minimax bounds for time and ensemble averages for the incompressible Navier-Stokes equations”**
- Joint work with Roger Temam (Indiana University)
- *Pure and Applied Functional Analysis (2022)*
- Dedicated to the memory of Ciprian Foias (20 July 1933 – 22 March 2020)

Inspired by

- **"Optimal bounds and extremal trajectories for time averages in nonlinear dynamical systems"**
- I. Tobasco, D. Goluskin, and C. R. Doering
- *Physics Letters A*, Volume 382 (2018), no. 6, 382–386.
- Charlie Doering (7 January 1956 – 15 May 2021)



Related to

- **Ergodic Optimization**
- Starting in the 1990s
  - "On the minimizing measures of Lagrangian dynamical systems", Ricardo Manẽ (1992)
- Main result
  - X compact
  - $T : X \to X$ continuous
  - $f : X \to \mathbb{R}$ continuous
  - Then
$$\sup_{\mu \in \mathcal{M}_{T\text{-inv}}} \int f \,\mathrm{d}\mu = \inf_{h \in \mathcal{C}(X)} \sup_{x \in X} (f + h \circ T - h)\,.$$
- Some recent references:
  - "Ergodic optimization in dynamical systems", O. Jenkinson (2019)
  - "Ergodic optimization of Birkhoff averages and Lyapunov exponents", Jairo Bochi (2017)

Estimates

Estimates are important for various purposes:

- **Assessing real quantities:** energy, drag coefficient, mechanical stress, heat transfer, chemical concentration, infected population, pharmaceutical dosage, etc.
- **Fundamental properties:** existence and uniqueness of solutions, global existence or blow-up, regularity
- **Control and stabilization results**
- **Dynamics:** existence, localization and dimension estimates of invariant sets, inertial manifolds, and global attractors

Instantaneous quantities - I

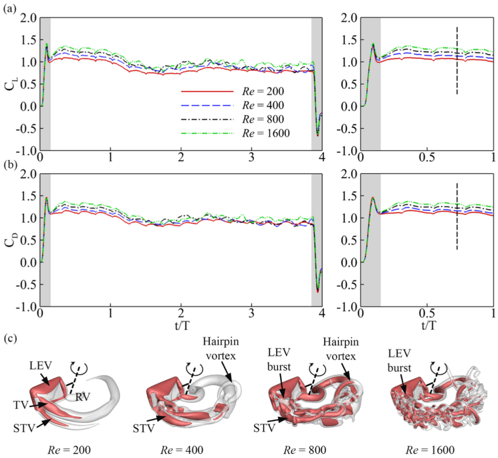


FIG. 7. The time history of lift (a) and drag (b) coefficients and wake structures (c) for *AR* = 2 rectangular wings at various Reynolds numbers (*Re*) with a fixed angle of attack of 45°. The shaded region indicates the acceleration/deceleration phases. The plots on the right side of (a) and (b) magnify the early-time shown in the left-side plots. The black dashed line denotes the time instant of the wake structures shown in plot (c). The gray shaded area indicates the period of the acceleration/deceleration phase.

Phys. Fluids **32**, 021905 (2020); doi: 10.1063/1.5134689 **32**, 021905-6

Instantaneous quantities - II

X.K. Wang et al. / Journal of Fluids and Structures 34 (2012) 51–67

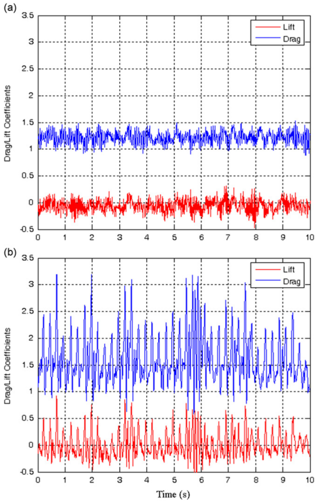
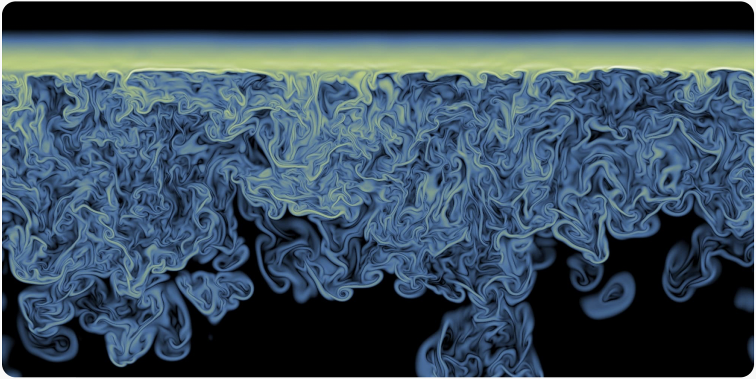


Fig. 10. Time history of the instantaneous drag and lift coefficients for $m^*=0.61$ at (a) Re=8000 and (b) 12 000.

Instantaneous quantities - III



Example of turbulent motion: Mixing at the cloud boundaries play a fundamental role in the evolution of clouds, and clouds can have in turn a profound impact on planetary scale circulations. This image depicts the turbulent structure of the stratocumulus cloud-top inside a vertical plane in terms of the magnitude of the temperature gradient, resolving scales from 4 meters down to about 4 milimeters. The upper horizontal stripe corresponds to the inversion that separates the turbulent cloud below from the warm clear sky above. The turbulent motion is created by the evaporation of the droplets in a thin region next to that inversion, which cools locally the fluid mixture and leads to finger structures plummeting into the cloud.(© Juan Pedro Mellado, Max Planck Institute for Meteorology).

Instantaneous versus averaged quantities

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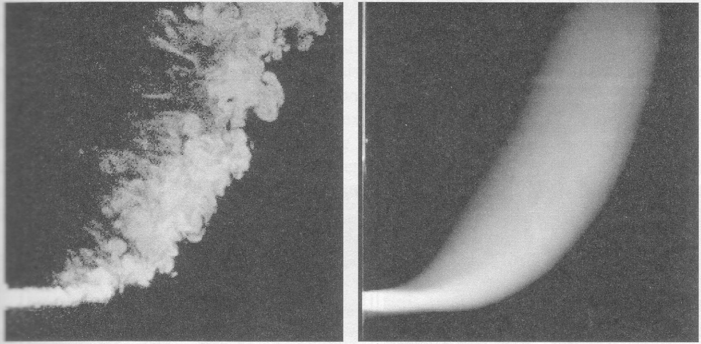
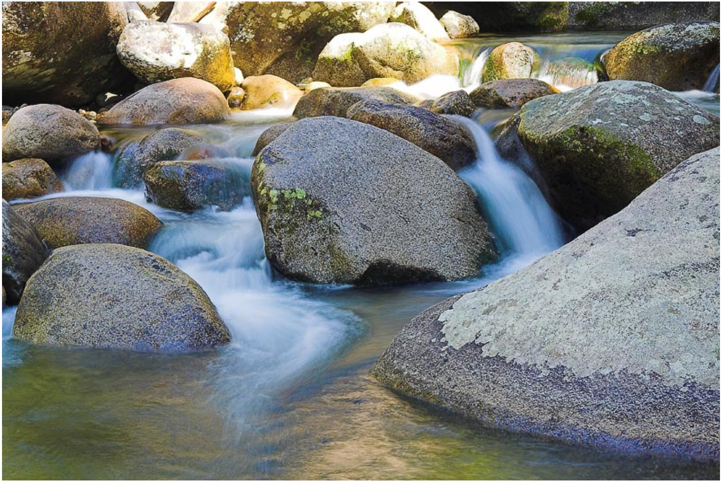


Figure 1.3 *Instantaneous and time averaged views of a jet in cross flow. The jet exits from the wall at left into a stream flowing from bottom to top (Su & Mungal, 1999).*

Averaged quantities



Asymptotic bounds

Want to bound

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T \phi(u(t))\,\mathrm{d}t, \quad \text{for} \quad \frac{\mathrm{d}u}{\mathrm{d}t} = F(u),$$

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Usual techniques:

- Energy-type estimates
- Variational inequalities

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More recently:

- Mini-max estimates...
- ... theoretically gives optimal results!
- ... numerically gives sharp results, often with less cost!

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Numerical approaches:

- Monte-Carlo estimates for classical time-evolution methods
- Linear programming/sums of squares for mini-max estimate

Differential equation

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In theory, can use energy-type estimates and so-on
Numerically, can use Monte-Carlo (solve multiple solutions for long time intervals, using a decent integration method, ...)
Or, **work only on the phase space, without solving the equation...**

Auxilliary function

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Now, if we find an **auxilliary function** V such that

$$\phi(u) + F(u) \cdot \nabla V(u) \leq C, \quad \forall u \in B,$$

then

$$\overline{\phi + F \cdot \nabla V} \leq C, \quad \text{hence} \quad \bar{\phi} \leq C.$$

Convex optimization problem

The problem

Find V s.th. $\phi(u) + F(u) \cdot \nabla V(u) \leq C, \ \forall u \in B,$ for *smallest possible* C,

can be written as a convex optimization problem

$$\sup \bar{\phi} \leq \inf_{(C,V) \in \mathbb{R} \times \mathcal{C}^1, \ S_{C,V}(u) \geq 0} C,$$

where $S_{C,V}(u) = C - \phi(u) - F(u) \cdot \nabla V(u)$.

(Optimization of linear map $(C, V) \mapsto C$ over a convex set, since $\mathbb{R} \times \mathcal{C}^1$ is convex and $S_{C,V}(u)$ is linear in C and V.)

Optimization with Sum of Squares (SoS)

If ϕ, F **polynomials**, we narrow minimization over **polynomials** V such that

$$S_{C,V}(u) = C - \phi(u) - F(u) \cdot \nabla V(u) = \texttt{SoS},$$

where $\texttt{SoS} = \sum_i p_i(u)^2$, for other polynomials p_i.

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Reducing the set of possible V's may increase the upper bound estimate, but at the advantage of turning it from NP-hard into a P-complete problem.

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Question: How large is the set of $\texttt{SoS}$?

Historical background on Sum of Squares

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- Hilbert's 17th problem (1900) is about decomposing positive polynomials into SoS of rational functionals.
- Hilbert had already noticed not all positive polynomials are SoSs.
- Reznik (2000) - survey of SoS and Hilbert's 17th problem
- Shor 1980s, 1990s, Choi, Lam, Reznik 1990s - SoS polynomial decomposition
- Parrilo (2000s) - several applications: Lyapunov functions, control, etc.
- Several SoS MATLAB toolbox solvers (2000s)
- Papachristodoulou, Peet (2006) - applications to PDEs
- Yu, Kashima, Imura (2008)- local stability of 2D fluid flows
- Goulart, Chernyshenko (2012) - global stability of fluid flows
- Fantuzzi, Goluskin, Doering, Goulart, Chernyshenko, Huang, Papachristodoulou (2010s) ...

Bounds for the van der Pol limit cycle

From "**Bounds for Deterministic and Stochastic Dynamical Systems using Sum-of-Squares Optimization**", by G. Fantuzzi, D. Goluskin, D. Huang, and S. I. Chernyshenko, in *SIAM J. Applied Dynamical Systems*, Vol. 15, No. 4, pp. 1962–1988.

$$\ddot{x} = \mu(1 - x^2)\dot{x} - x$$

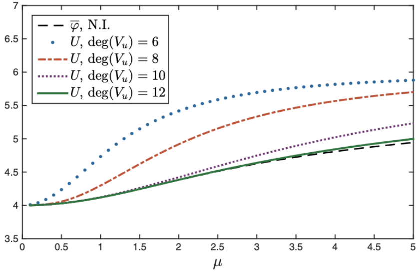
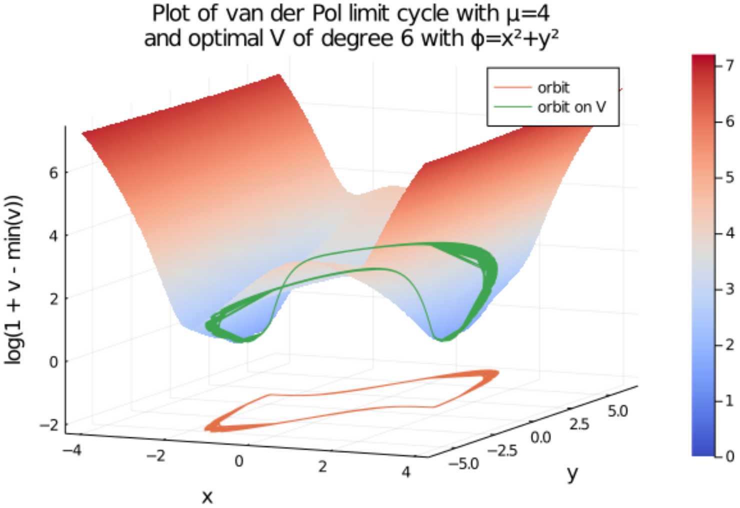


Figure 2. *Optimal upper bounds on $\overline{\varphi} = \overline{x^2 + y^2}$ for the van der Pol oscillator computed with the upper bound problem of (2.9) for different degrees of V_u. The time average $\overline{\varphi}$ obtained by numerical integration (N.I.) of the system is also shown.*

Auxiliary function with degree 6 for van der Pol



Auxiliary function with degree 8 for van der Pol

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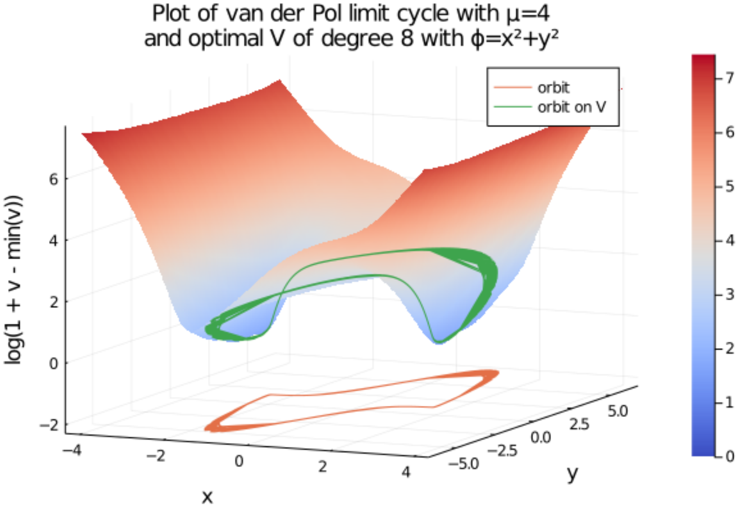

Auxiliary function with degree 10 for van der Pol

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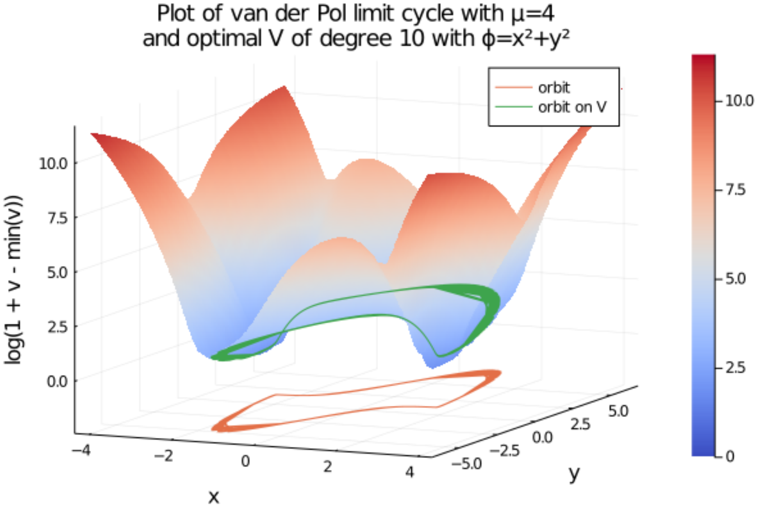

Minimax problem

The problem

Find V and the *smallest possible* C s.th. $\phi(u)+F(u)\cdot\nabla V(u)\leq C,\ \forall u\in B,$

can also be written as the minimax problem

$$\sup_{u_0\in B}\bar{\phi}(u_0)\leq\min_{V\in\mathcal{C}^1(B)}\max_{u\in B}\left\{\phi(u)+F(u)\cdot\nabla V(u)\right\}.$$

Optimality of the minimax formula

Tobasco-Goluskin-Doering (2018):

It turns out, the minimax formula is optimal and is achieved!

$$\max_{u_0 \in B} \bar{\phi}(u_0) = \min_{V \in \mathcal{C}^1(B)} \max_{u \in B} \left\{ \phi(u) + F(u) \cdot \nabla V(u) \right\}.$$

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Conditions:

- $X = \mathbb{R}^n$
- $F : \mathbb{R}^n \to \mathbb{R}^n$ continuously differentiable
- $B \subset \mathbb{R}^n$ compact
- B positively invariant
- $\phi : B \to \mathbb{R}$ continuous

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Questions:

- What about infinite-dimensional systems? Like 2D NSE.
- What about 3DNSE?

Proof of optimality in the finite-dimensional case

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$$\max_{u_0 \in B} \bar{\phi}(u_0) = \max_{u_0 \in B} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \phi(u(t)) \, \mathrm{d}t \text{ (definition)}$$

Proof of optimality in the finite-dimensional case

Tobasco-Goluskin-Doering (2018):

$$
\begin{aligned}
\max_{u_0 \in B} \bar{\phi}(u_0) &= \max_{u_0 \in B} \limsup_{T \to \infty} \frac{1}{T} \int_0^T \phi(u(t)) \,\mathrm{d}t \\
&= \max_{\mu \in \mathcal{P}_{\mathrm{inv}}(B)} \int_B \phi \,\mathrm{d}\mu \text{ (maximum at extreme, ergodic meas.)}
\end{aligned}
$$

Proof of optimality in the finite-dimensional case

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&= \max_{\mu\in\mathcal{P}_{\mathcal{E}\text{-}\mathrm{inv}}(B)} \int_B \phi\,\mathrm{d}\mu \\
&= \sup_{\mu\in\mathcal{P}(B)} \inf_{V\in\mathcal{C}^1(B)} \int_B \phi + F\cdot\nabla V\,\mathrm{d}\mu \quad \left(\inf_V \langle F\cdot\nabla V\rangle = \begin{cases} 0, & \mu\in\mathcal{P}_{\mathcal{E}\text{-}\mathrm{inv}}(B) \\ -\infty, & \mu\notin\mathcal{P}_{\mathcal{E}\text{-}\mathrm{inv}}(B) \end{cases} \right)
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&= \sup_{\mu \in \mathcal{P}(B)} \inf_{V \in \mathcal{C}^1(B)} \int_B \phi + F \cdot \nabla V\,\mathrm{d}\mu \\
&= \inf_{V \in \mathcal{C}^1(B)} \sup_{\mu \in \mathcal{P}(B)} \int_B \phi + F \cdot \nabla V\,\mathrm{d}\mu \quad \text{(minimax principle)}
\end{aligned}
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&= \max_{\mu \in \mathcal{P}_{\mathcal{E}\text{-}\mathrm{inv}}(B)} \int_B \phi\,\mathrm{d}\mu \\
&= \sup_{\mu \in \mathcal{P}(B)} \inf_{V \in \mathcal{C}^1(B)} \int_B \phi + F\cdot\nabla V\,\mathrm{d}\mu \\
&= \inf_{V \in \mathcal{C}^1(B)} \sup_{\mu \in \mathcal{P}(B)} \int_B \phi + F\cdot\nabla V\,\mathrm{d}\mu \\
&= \inf_{V \in \mathcal{C}^1(B)} \max_{u \in B} \{\phi(u) + F(u)\cdot\nabla V(u)\} \quad \text{(extreme at Dirac delta)}
\end{aligned}
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2D NSE

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2D NSE

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$$= \max_{\mu \in \mathcal{P}_{\mathrm{inv}}(B)} \int_B \phi \,\mathrm{d}\mu \text{ (maximum at extreme, ergodic meas.)}$$

2D NSE

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&= \max_{\mu\in\mathcal{P}_{\mathrm{inv}}(B\cap K)} \int_{B\cap K} \phi\,\mathrm{d}\mu \quad (K \text{ compact attracting; } B \text{ normal})
\end{aligned}
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&= \max_{\mu\in\mathcal{P}_{\mathrm{wsss}}(B\cap K)} \int_{B\cap K} \phi\,\mathrm{d}\mu \;\; (\mathcal{P}_{\mathrm{inv}}(B\cap K) = \mathcal{P}_{\mathrm{sss}}(B\cap K) = \mathcal{P}_{\mathrm{wsss}}(B\cap K))
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&= \sup_{\mu\in\mathcal{P}(B\cap K)} \inf_{V\in\mathcal{T}_{\mathsf{cyl}}} \int_{B\cap K} \phi + F\cdot\nabla V\,\mathrm{d}\mu \quad \left(\inf_V = \begin{cases} 0, & \mu\in\mathcal{P}_{\mathsf{wsss}}(B\cap K) \\ -\infty, & \mu\notin\mathcal{P}_{\mathsf{wsss}}(B\cap K)\end{cases}\right)
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2D NSE

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&\leq \max_{\mu\in\mathcal{P}_{\text{wsss}}(B)} \int_{B\cap K} \phi\,\mathrm{d}\mu \text{ (Bogoliubov-Krylov; } K \text{ comp. attr.; } B \text{ normal)}
\end{aligned}
$$

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\end{aligned}
$$

Lagrangian vs Eulerian invariance

- Each u_0: $\displaystyle\int_0^T F(S(t)u_0)\cdot\nabla V(S(t)u_0)\,\mathrm{d}t = V(S(T)u_0) - V(u_0)$
- Ensemble averaging

$$\int_0^T \int_X F(S(t)u_0)\cdot\nabla V(S(t)u_0)\,\mathrm{d}\mu(u_0)\,\mathrm{d}t = \int_X V(S(T)u_0)\,\mathrm{d}\mu(u_0) - \int_X V(u_0)\,\mathrm{d}\mu(u_0)$$

- Equivalences

$$\begin{aligned} S(T)\mu = \mu &\Leftrightarrow \int_X V(S(T)u_0)\,\mathrm{d}\mu(u_0) = \int_X V(u_0)\,\mathrm{d}\mu(u_0) \\ &\Leftrightarrow \int_X F(S(t)u_0)\cdot\nabla V(S(t)u_0)\,\mathrm{d}\mu(u_0) = 0 \\ &\Leftrightarrow \int_X F(v_0)\cdot\nabla(V\circ S(t))(v_0)\,\mathrm{d}\mu(v_0) = 0. \end{aligned}$$

Concepts for PDEs: test functionals

X Hausdorff; V, W Banach; $W \subset V \subset X \subset V' \subset W'$ continuous.

Definition

A **cylindrical test functional** on W' is $\Psi : W' \to \mathbb{R}$ of the form

$$\Psi(u) = \psi(\langle u, w_1\rangle_{W',W}, \ldots, \langle u, w_m\rangle_{W',W}), \qquad \forall u \in W',$$

where $w_1, \ldots, w_m \in W$, $m \in \mathbb{N}$, and $\psi \in \mathcal{C}_c^1(\mathbb{R}^m)$. Space $\mathcal{T}^{\mathrm{cyl}}(W')$.

Definition

A **general test functional** for the Navier-Stokes equations is a continuous functional $\Psi : V \to \mathbb{R}$ which is Fréchet H-differentiable in the direction of V and with $\mathrm{u} \mapsto D_{\mathrm{u}}\Psi(\mathrm{u})$ continuous and bounded from V into V.

Concepts for PDEs: stationary statistical solutions

$$u_t = F(u), \qquad F : V \to V', \quad F : X \to W'$$

Definition

A **weak stationary statistical solution** is a Borel probability measure μ on X such that, for any $\mathcal{T}^{\mathrm{cyl}}(W')$, the map $u \mapsto \langle F(u), \Psi'(u)\rangle_{W',W}$ is μ-integrable and

$$\int_X \langle F(u), \Psi'(u)\rangle_{W',W} \,\mathrm{d}\mu(v) = 0.$$

Denote by $\mathcal{P}_{\mathrm{wsss}}(E)$ the space of weak stationary statistical solution carried by a Borel subset $E \subset X$.
Denote by $\mathcal{P}_{\mathrm{sss}}(E) \subset \mathcal{P}_{\mathrm{wsss}}(E)$ the subspace of those for which the equation holds for all general test functionals.

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Theorem

Let B be a positively invariant set for $\{S(t)\}_{t\geq 0}$ which is closed in X and normal, and suppose that there exists a compact and metrizable subset K of X which attracts the points of B. Suppose F is continuous on K. Let $\phi \in \mathcal{C}_b(B)$. Then,

$$\max_{\mu\in\mathcal{P}_{\mathbf{wsss}}(K)} \int_K \phi(u)\;d\mu(u) = \inf_{\Psi\in\mathcal{T}_{cyl}(W')}\max_{u\in B\cap K}\left\{\phi(u) + \langle F(u), \Psi'(u)\rangle_{W',W}\right\}.$$

Suppose, further, that $\mathcal{P}_{\mathbf{wsss}}(K) = \mathcal{P}_{inv}(B\cap K)$. Then

$$\max_{u_0\in B}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \phi(S(t)u_0)\;dt = \inf_{\Psi\in\mathcal{T}_{cyl}(W')}\max_{u\in B\cap K}\left\{\phi(u) + \langle F(u), \Psi'(u)\rangle_{W',W}\right\}.$$

Thm II - 2D NSE

Theorem

For the 2D NSE, let $\{S(t)\}_{t\geq 0}$ *be the associated continuous semigroup and let* $\phi \in \mathcal{C}_b(H)$. *Then,*

$$
\max_{u_0\in B} \limsup_{T\to\infty} \frac{1}{T}\int_0^T \phi(S(t)u_0)\,dt = \inf_{\Psi\in\mathcal{T}_{cyl}(W')} \max_{u\in B_V} \left\{\phi(u) + \langle F(u), \Psi'(u)\rangle_{W',W}\right\},
$$

where $W = D(A)\cap W^{1,\infty}(\Omega)^2$ *and* B_V *is an absorbing closed ball in* V.

Thm III - 3D NSE

Theorem

For the 3D NSE, with $X = H_w$, $W = D(A)$, and $K \supset \mathcal{A}_w$ compact, $B \subset H$ bounded and positively invariant, $\phi \in \mathcal{C}_b(B)$,

$$\sup_{u \in \mathcal{U}(B)} \limsup_{T\to\infty} \frac{1}{T}\int_0^T \phi(S(t)u_0)\,dt$$
$$\leq \sup_{\mu \in \mathcal{P}_{fpsss}(K)} \int_K \phi(u)\,d\mu(u)$$
$$\leq \inf_{\Psi \in \mathcal{T}_{cyl}(W')} \max_{u \in B\cap K} \left\{\phi(u) + \langle F(u), \Psi'(u)\rangle_{W',W}\right\}.$$