Convergence of the Euler method for Random ODEs driven by semi-martingales noises

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Joint work with Peter Kloeden (University of Tübingen, Germany)

Seminário de Probabilidade - IM-UFRJ

June 8, 2026

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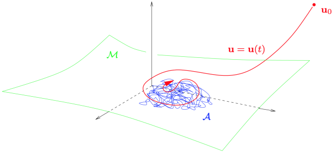
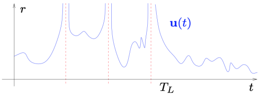
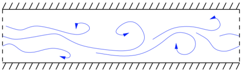
Research areas (essentially deterministic)

- PDEs
- Infinite-dimensional dynamics
- Attractors, inertial manifolds, etc.
- Navier-Stokes equations
- Statistical solutions (random initial data, measure transport)
- Turbulence







Recent stochastic interest

- Stochastic models in epidemics (during the COVID-19 pandemic)
- Stochastic models in applied math in general
- Numerical approximations of Random ODEs
- SDE-based AI generative methods in the assessment of reservoir properties and ocean current predictions with ExxonMobil
- Stochastic Navier-Stokes equations for turbulence modeling

Current PDE works with stochastic counterparts

- Optimal minimax bounds for ensemble averages of the Navier-Stokes equations

$$\max_{\substack{\mu \text{ stationary} \\ \mathrm{supp}(\mu)\subset K}} \int_K \phi(u)\,\mathrm{d}\mu(u) = \inf_{\substack{\Psi \text{ cyl.} \\ \text{test fn}}} \max_{u\in K}\{\phi(u) + \langle F(u), \Psi'(u)\rangle\}.$$

for $\frac{\mathrm{d}u}{\mathrm{d}t} = F(u)$ on phase space X; $K \subset X$ compact; $\phi \in \mathcal{C}(K)$:

- Conditions for the existence of energy and enstrophy cascades in 3D and 2D Navier-Stokes equations
- Other stochastic Navier-Stokes problems (zero-noise limit, etc.)

Today

- *"Strong order-one convergence of the Euler method for random ordinary differential equations driven by semi-martingale noises"*
- Joint work with Peter Kloeden (University of Tübingen, Germany)
- ESAIM: M2AN, Volume 59, Number 6, November-December 2025
- DOI: 10.1051/m2an/2025087
- github rmsrosa/rode_conv_em (reproducible Julia code)

Strong order of convergence

Strong convergence means pathwise convergence for random variables but it means convergence in mean for approximations of SDEs and RODEs:

- Consider a stochastic process $\{X_t\}_t$
- Consider an approximation X_j^N of X_{t_j} at $t_j = j\Delta t_N$
- Method is of *pathwise* order p if there exists $C = C(\omega)$ finite almost surely such that

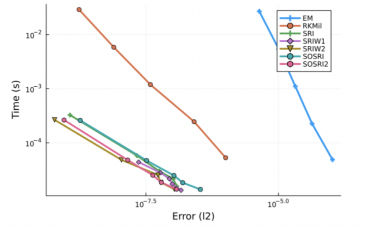
$$\max_j \|X_{t_j}(\omega) - X_j^N(\omega)\| \leq C(\omega)\Delta t_N^p$$

- Method is of *strong* order p if there exists a constant C such that

$$\max_j \mathbb{E}[\|X_{t_j} - X_j^N\|] \leq C\Delta t_N^p$$

What is the story about that?

- Teaching theory and numerics for SDEs and RODEs (since 2022/1)
- Illustrating the efficiency of different methods (SciML - Julia lang)



- Wanted to illustrate strong order of convergence:
  - strong order 1/2 of Euler-Maruyama for SDEs
  - strong order θ of Euler for RODEs
- Ok for SDEs
- Couldn't find example for RODEs

What is a RODE?

Types of differential equations

- ODE:
$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(t, x);$$
- SDE (Itô diffusion):
$$\mathrm{d}X_t = f(t, X_t)\,\mathrm{d}t + g(t, X_t)\,\mathrm{d}W_t;$$
- RODE:
$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = f(t, X_t, Y_t);$$
- Extra driving terms
  - SDE: white noise "$\mathrm{d}W_t/\mathrm{d}t$" (specific and very irregular),
  - RODE: noise process Y_t ("arbitrary" but not so irregular).

Why RODEs?

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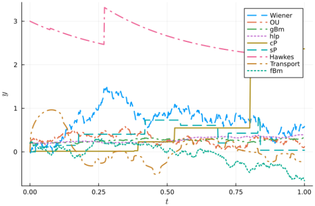
- When noise is not just an Itô noise... and can be computed exactly, e.g.
  - Ornstein-Uhlenbeck (colored noise) process
  - geometric Brownian motion
  - fractional Brownian motion
  - Transport process
  - Point process (Poisson, Hawkes, etc.)
  - time-changed Wiener process



  - (Some can be embedded into a system of SDEs but not all)

Ornstein-Uhlenbeck approximates white noise

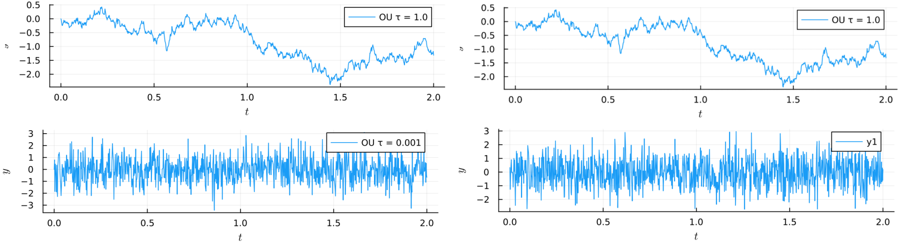
Ornstein-Uhlenbeck

$$\tau \mathrm{d}O_t = -dt + \zeta \, \mathrm{d}W_t$$

with statistics

$$\mathbb{E}[O_t] = O_0 e^{-\frac{t}{\tau}}, \quad \mathrm{Var}(O_t) = \frac{\varsigma^2}{2\tau}, \quad \mathrm{Cov}(O_t, O_s) = \frac{\varsigma^2}{2\tau} e^{-\frac{|t-s|}{\tau}}.$$

approximates a white noise as time scale $\tau \to 0$ and $\zeta \to 1$ (or $\nu, \sigma \to \infty$, $\nu/\sigma \to 1$ in the usual $\nu = 1/\tau$ and $\sigma = \zeta/\tau$).



E.g. population growth with random perturbation

- ODE: deterministic specific growth

$$\mu_t = \mu \qquad \Longrightarrow \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = \mu x$$

- SDE: growth perturbed by white noise

$$\mu_t = \mu + \sigma \frac{\mathrm{d}W_t}{\mathrm{d}t} \qquad \Longrightarrow \qquad \mathrm{d}X_t = \mu X_t \,\mathrm{d}t + \sigma X_t \,\mathrm{d}W_t$$

- RODE: growth perturbed by colored OU noise

$$\mu_t = \mu + \sigma O_t \qquad \Longrightarrow \qquad \frac{\mathrm{d}X_t}{\mathrm{d}t} = (\mu + \sigma O_t) X_t$$

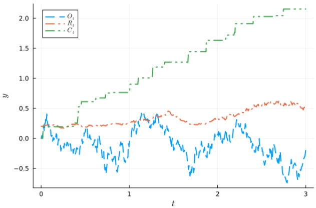
Actuarial risk model

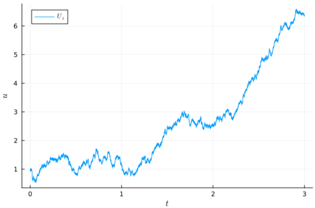
- Surplus $U_t = U_0 + \gamma t - \sum_{i=1}^{N_t} C_i$; claims $C_t = \sum_{i=1}^{N_t} C_i$, premium γ:
- Jump differential equation formulation: $\mathrm{d}U_t = \gamma\,\mathrm{d}t - \mathrm{d}C_t$
- Premium perturbed by white noise + stochastic interest rate R_t

$$\mathrm{d}U_t = (\gamma + R_t U_t)\,\mathrm{d}t + \varepsilon\,\mathrm{d}W_t - \mathrm{d}C_t.$$

- Write $X_t = U_t - C_t - O_t$ with OU process $\mathrm{d}O_t = -\nu O_t\,\mathrm{d}t + \varepsilon\,\mathrm{d}W_t$
- Get RODE model (R_t being gBm or Vasicek)

$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = R_t X_t + R_t(C_t + O_t) + \nu O_t + \gamma.$$

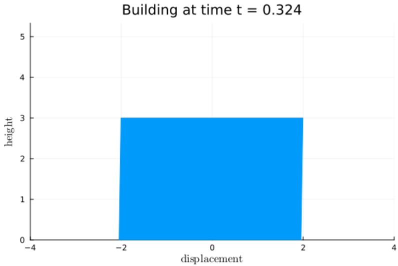


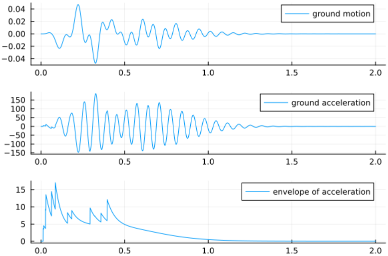


Earthquake ground-shaking model

- Mechanical structure (ceiling of a one-storey building)

$$\begin{cases} \ddot{X}_t + 2\zeta_0\omega_0\dot{X}_t + \omega_0^2 X_t = -\ddot{M}_t \\ X_0 = 0, \quad \dot{X}_0 = 0 \end{cases}$$





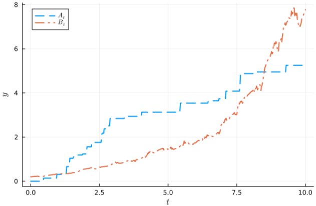
- Ground motion $M_t = \sum_{i=1}^k \gamma_i (t-\tau_i)_+^2 e^{-\delta_i(t-\tau_i)} \cos(\omega_i(t-\tau_i))$,
- (Or with $\ddot{M}_t$ white noite, Hawkes process, etc.)

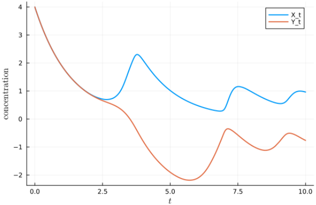
Toggle-switch gene-expression model

- X_t and Y_t are protein products of interacting genes

$$
\begin{cases}
\dfrac{\mathrm{d}X_t}{\mathrm{d}t} = \left(A_t + \dfrac{X_t^4}{a^4 + X_t^4}\right)\left(\dfrac{b^4}{b^4 + Y_t^4}\right) - \mu X_t, \\
\dfrac{\mathrm{d}Y_t}{\mathrm{d}t} = \left(B_t + \dfrac{Y_t^4}{c^4 + Y_t^4}\right)\left(\dfrac{d^4}{d^4 + X_t^4}\right) - \nu Y_t,
\end{cases}
$$

- a, b, c, d threshold parameters; μ, λ decay rates
- External (noise) activation parameters $\{A_t\}_{t\geq 0}$ and $\{B_t\}_{t\geq 0}$ (e.g. Compound Poisson and GBm)



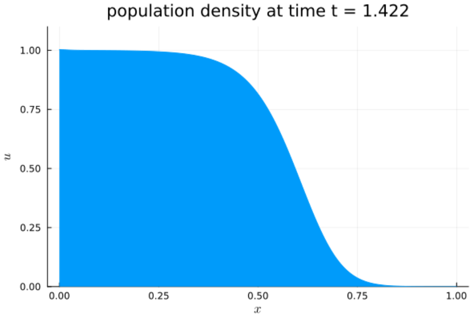


Random Fisher-KPP PDE

- $u = u(t, x)$ population density

$$\begin{cases} \dfrac{\partial u}{\partial t} = \mu \dfrac{\partial^2 u}{\partial x^2} + \lambda u \left(1 - \dfrac{u}{u_m}\right), & (t, x) \in (0, \infty) \times (0, 1), \\ \dfrac{\partial u}{\partial x}(t, 0) = -Y_t, \quad \dfrac{\partial u}{\partial x}(t, 1) = 0, \end{cases}$$

- random incoming migrations Y_t as a colored OU noise modulated by an exponentially decaying Hawkes process



Convergence of the Euler method

Known before:

- ODE: typically order 1
- SDE (additive noise): order 1
- SDE (multiplicative noise): order 1/2
- RODE with θ-Hölder noise: order θ

Remarks:

- ODE is order 1 for smooth RHS $f(t, x)$
- ODE is *at least* order θ for f θ-Hölder in time
- But noises have more structure than just pathwise Hölder regularity...

Looking for an example

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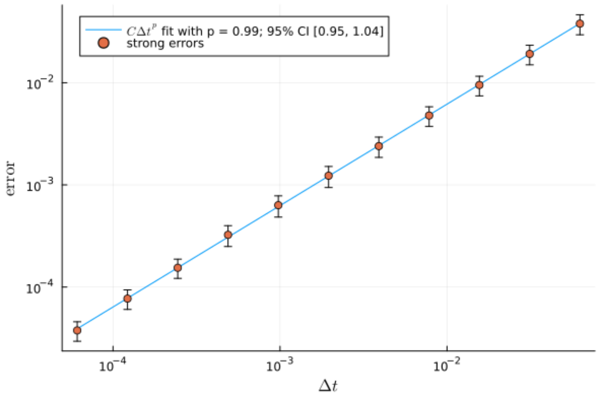
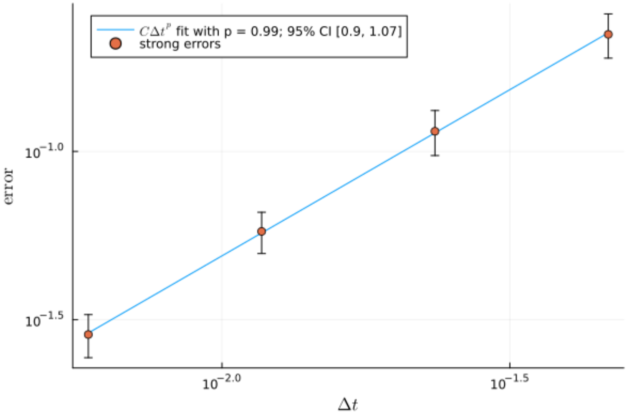


- Example

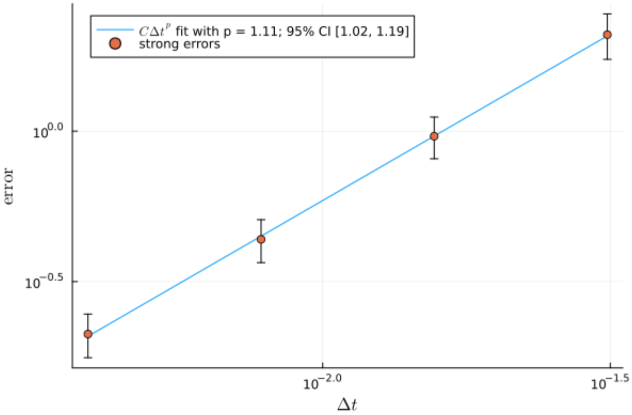
$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = W_t X_t$$

- Same with a few other equations and noises...

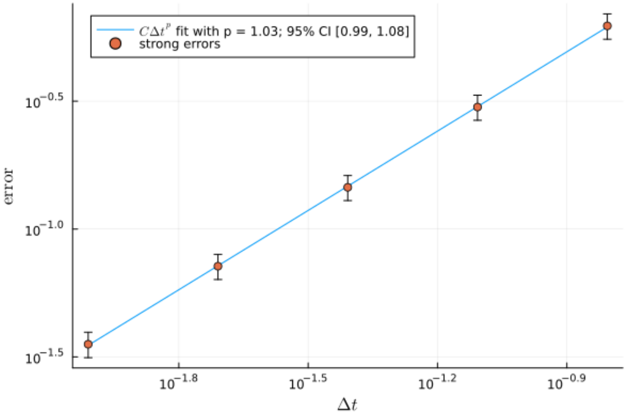
Order of convergence - risk model



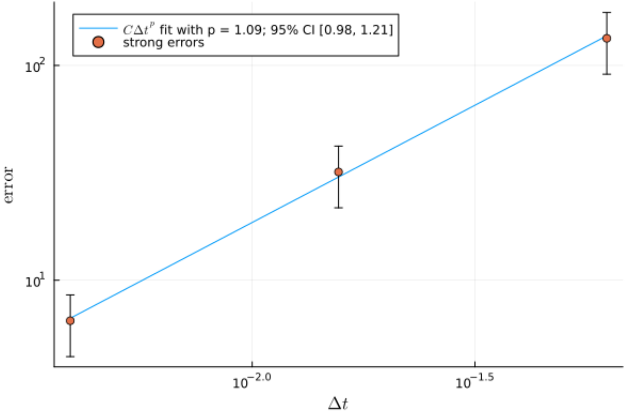
Order of convergence - Earthquake



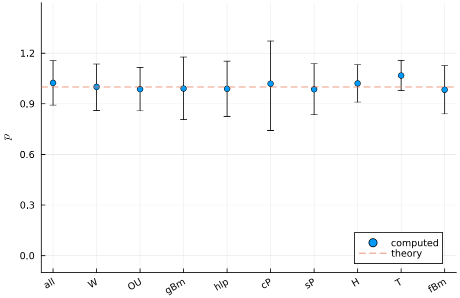
Order of convergence - toggle switch model

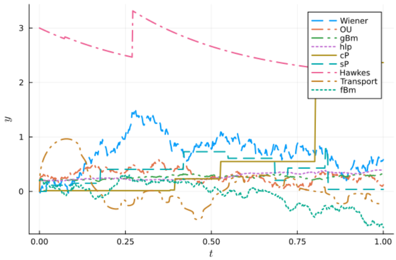


Order of convergence - random Fisher-KPP



Order of convergence - linear with “all” noises

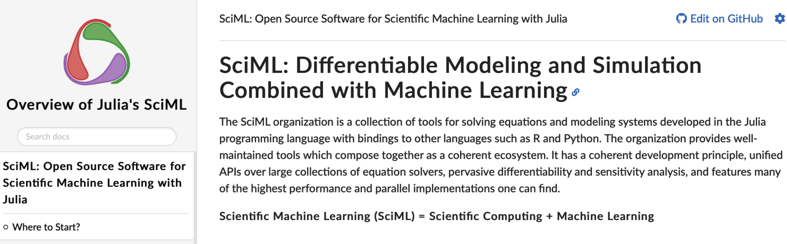




$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = \|Y_t\|^2 X_t + Y_t$$

Looking for help...

SciML's julia language community


Overview of Julia's SciML

Search docs

SciML: Open Source Software for Scientific Machine Learning with Julia

- Where to Start?

SciML: Open Source Software for Scientific Machine Learning with Julia   Edit on GitHub

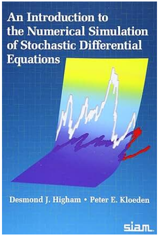
SciML: Differentiable Modeling and Simulation Combined with Machine Learning

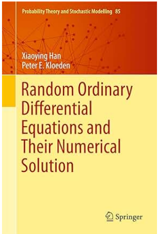
The SciML organization is a collection of tools for solving equations and modeling systems developed in the Julia programming language with bindings to other languages such as R and Python. The organization provides well-maintained tools which compose together as a coherent ecosystem. It has a coherent development principle, unified APIs over large collections of equation solvers, pervasive differentiability and sensitivity analysis, and features many of the highest performance and parallel implementations one can find.

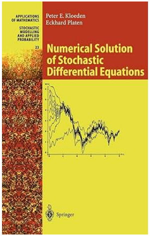
Scientific Machine Learning (SciML) = Scientific Computing + Machine Learning

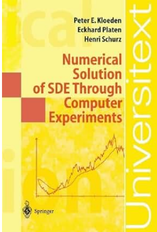
Looking for help...

Peter Kloeden









Fractional Brownian motion noise

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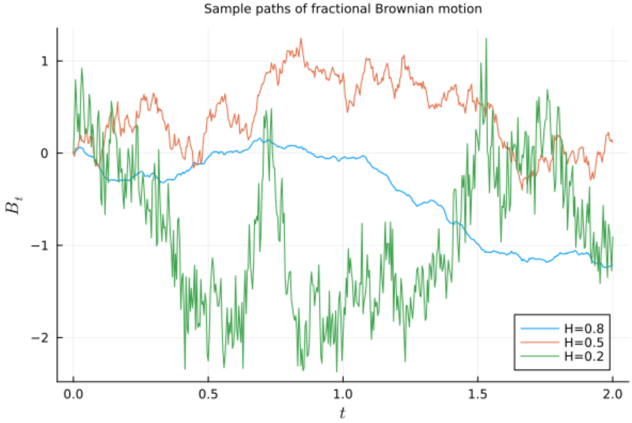
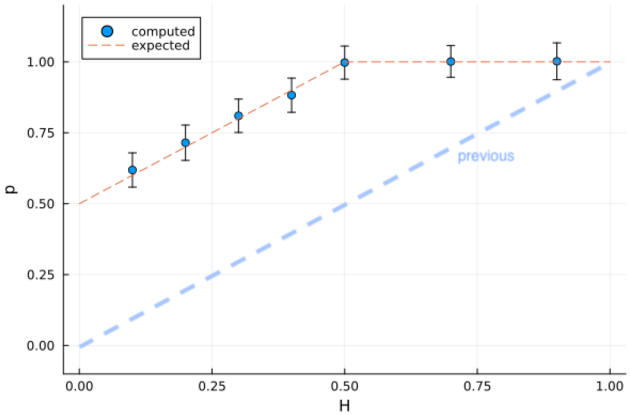
- Covariance $\mathbb{E}[B_t^{(H)} B_s^{(H)}] = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right)$
  - $0 < H < 1/2$: steps are negatively correlated (rougher)
  - $H = 1/2$: steps are not correlated (Wiener process)
  - $1/2 < H < 1$: steps are positively correlated (less rough)
  - Pathwise Hölder regularity $|B_t^{(H)} - B_s^{(H)}| \leq c|t-s|^{H-\varepsilon}$



Order of convergence - linear equation with fBm



$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = -X_t + B_t^{(H)}$$

Order of convergence - RODE

- Consider solution $\{X_t\}_t$ of RODE

$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = f(t, X_t, Y_t)$$

- Approximation X_j^N of X_{t_j} at $t_j = j\Delta t_N$,

$$X_{j+1}^N = X_j^N + f(t_j, X_j^N, Y_{t_j})\Delta t,$$

- Method is of *pathwise* order p if there exists $C = C(\omega)$ finite almost surely such that

$$\max_j \|X_{t_j}(\omega) - X_j^N(\omega)\| \leq C(\omega)\Delta t_N^p$$

- Method is of *strong* order p if there exists a constant C such that

$$\max_j \mathbb{E}[\|X_{t_j} - X_j^N\|] \leq C\Delta t_N^p$$

Euler for RODEs

- $$\frac{\mathrm{d}X_t}{\mathrm{d}t} = f(t, X_t, Y_t)$$

Euler for RODEs

- $$\frac{\mathrm{d}X_t}{\mathrm{d}t} = f(t, X_t, Y_t)$$
- $$X_{t+\Delta t} = X_t + \int_t^{t+\Delta t} f(s, X_s, Y_s)\,\mathrm{d}s$$

Euler for RODEs

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- $X_{t_j+\Delta t} = X_{t_j} + f(t_j, X_{t_j}, Y_{t_j})\Delta t$
- $\Rightarrow \epsilon_j = \displaystyle\int_{t_j}^{t_j+\Delta t} (f(s, X_s, Y_s) - f(t_j, X_{t_j}, Y_{t_j}))\,\mathrm{d}s$

Euler for RODEs

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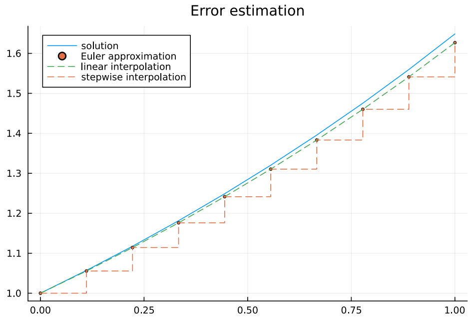

Euler for RODEs - Hölder noise

- $\dfrac{\mathrm{d}X_t}{\mathrm{d}t} = f(t, X_t, Y_t)$

Euler for RODEs - Hölder noise

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Euler for RODEs - Hölder noise

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Euler for RODEs - Hölder noise

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- f and X_t Lipschitz...

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- f and X_t Lipschitz... $\Rightarrow \epsilon_j \leq \displaystyle\int_{t_j}^{t_j+\Delta t} \left(A|s - t_j| + B|Y_s - Y_{t_j}|\right)\mathrm{d}s$

Euler for RODEs - Hölder noise

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- $\dfrac{\mathrm{d}X_t}{\mathrm{d}t} = f(t, X_t, Y_t)$
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- f and X_t Lipschitz... $\Rightarrow \epsilon_j \leq \displaystyle\int_{t_j}^{t_j+\Delta t} (A|s - t_j| + B|Y_s - Y_{t_j}|)\,\mathrm{d}s$
- BUT $\{Y_t\}_t$ is θ-Hölder

Euler for RODEs - Hölder noise

- $\dfrac{\mathrm{d}X_t}{\mathrm{d}t} = f(t, X_t, Y_t)$
- $X_{t+\Delta t} = X_t + \displaystyle\int_t^{t+\Delta t} f(s, X_s, Y_s)\,\mathrm{d}s$
- $X_{t_j+\Delta t} = X_{t_j} + f(t_j, X_{t_j}, Y_{t_j})\Delta t$
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- BUT $\{Y_t\}_t$ is θ-Hölder $\Rightarrow \epsilon_j \lesssim \Delta t^{1+\theta}$

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$$\epsilon = \sum_j \epsilon_j \lesssim \sum_j \Delta t^{1+\theta} \lesssim O(\Delta t^{\theta}).$$

Euler for RODEs - Hölder noise

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- BUT $\{Y_t\}_t$ is θ-Hölder $\Rightarrow \epsilon_j \lesssim \Delta t^{1+\theta}$
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$$\epsilon = \sum_j \epsilon_j \lesssim \sum_j \Delta t^{1+\theta} \lesssim O(\Delta t^\theta).$$

- Hence strong convergence order θ (Wang, Cao, Han & Kloeden (2021))

Euler for RODEs - idea

- $\dfrac{\mathrm{d}X_t}{\mathrm{d}t} = f(t, X_t, Y_t).$

Euler for RODEs - idea

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Euler for RODEs - idea

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- $f(s, X_s, Y_s) - f(t_j(s), X_{t_j(s)}, Y_{t_j(s)}) = \displaystyle\int_{t_j(s)}^{s} \mathrm{d}F_s$

Euler for RODEs - idea

- $\frac{\mathrm{d}X_t}{\mathrm{d}t} = f(t, X_t, Y_t)$.
- $\Rightarrow \epsilon_j = \int_{t_j}^{t_j+\Delta t} (f(s, X_s, Y_s) - f(t_j, X_{t_j}, Y_{t_j}))\,\mathrm{d}s$
- $f(s, X_s, Y_s) - f(t_j(s), X_{t_j(s)}, Y_{t_j(s)}) = \int_{t_j(s)}^{s} \mathrm{d}F_s$
- Global error:

$$\epsilon = \int_0^t (f(s, X_s, Y_s) - f(t_j(s), X_{t_j(s)}, Y_{t_j(s)}))\,\mathrm{d}s = \int_0^t \int_{t_j(s)}^{s} \mathrm{d}F_\tau\,\mathrm{d}s$$

Euler for RODEs - idea

- $\dfrac{\mathrm{d}X_t}{\mathrm{d}t} = f(t, X_t, Y_t).$
- $\Rightarrow \epsilon_j = \displaystyle\int_{t_j}^{t_j+\Delta t} (f(s, X_s, Y_s) - f(t_j, X_{t_j}, Y_{t_j}))\,\mathrm{d}s$
- $f(s, X_s, Y_s) - f(t_j(s), X_{t_j(s)}, Y_{t_j(s)}) = \displaystyle\int_{t_j(s)}^{s} \mathrm{d}F_s$
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$$\epsilon = \int_0^t (f(s, X_s, Y_s) - f(t_j(s), X_{t_j(s)}, Y_{t_j(s)}))\,\mathrm{d}s = \int_0^t \int_{t_j(s)}^{s} \mathrm{d}F_\tau\,\mathrm{d}s$$

$$= \int_0^t \int_\tau^{t_j(\tau)+\Delta t} \mathrm{d}s\,\mathrm{d}F_\tau \text{ (using Fubini)}$$

Euler for RODEs - idea

- $\dfrac{\mathrm{d}X_t}{\mathrm{d}t} = f(t, X_t, Y_t).$
- $\Rightarrow \epsilon_j = \displaystyle\int_{t_j}^{t_j+\Delta t} (f(s, X_s, Y_s) - f(t_j, X_{t_j}, Y_{t_j}))\,\mathrm{d}s$
- $f(s, X_s, Y_s) - f(t_j(s), X_{t_j(s)}, Y_{t_j(s)}) = \displaystyle\int_{t_j(s)}^{s} \mathrm{d}F_s$
- Global error:

$$
\begin{aligned}
\epsilon &= \int_0^t (f(s, X_s, Y_s) - f(t_j(s), X_{t_j(s)}, Y_{t_j(s)}))\,\mathrm{d}s = \int_0^t \int_{t_j(s)}^{s} \mathrm{d}F_\tau\,\mathrm{d}s \\
&= \int_0^t \int_\tau^{t_j(\tau)+\Delta t} \mathrm{d}s\,\mathrm{d}F_\tau \ \text{(using Fubini)} \\
&= \int_0^t (t_j(\tau) + \Delta t - \tau)\,\mathrm{d}F_\tau
\end{aligned}
$$

Euler for RODEs - idea

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- $\dfrac{\mathrm{d}X_t}{\mathrm{d}t} = f(t, X_t, Y_t)$.
- $\Rightarrow \epsilon_j = \displaystyle\int_{t_j}^{t_j+\Delta t} (f(s, X_s, Y_s) - f(t_j, X_{t_j}, Y_{t_j}))\,\mathrm{d}s$
- $f(s, X_s, Y_s) - f(t_j(s), X_{t_j(s)}, Y_{t_j(s)}) = \displaystyle\int_{t_j(s)}^{s} \mathrm{d}F_s$
- Global error:

$$\begin{aligned}\epsilon &= \int_0^t (f(s, X_s, Y_s) - f(t_j(s), X_{t_j(s)}, Y_{t_j(s)}))\,\mathrm{d}s = \int_0^t \int_{t_j(s)}^{s} \mathrm{d}F_\tau\,\mathrm{d}s \\ &= \int_0^t \int_\tau^{t_j(\tau)+\Delta t} \mathrm{d}s\,\mathrm{d}F_\tau \text{ (using Fubini)} \\ &= \int_0^t (t_j(\tau) + \Delta t - \tau)\,\mathrm{d}F_\tau \leq C\Delta t \text{ (Chain rule, Itô formula, etc.)}\end{aligned}$$

Euler for RODEs - idea

- $\dfrac{\mathrm{d}X_t}{\mathrm{d}t} = f(t, X_t, Y_t)$.
- $\Rightarrow \epsilon_j = \displaystyle\int_{t_j}^{t_j+\Delta t} (f(s, X_s, Y_s) - f(t_j, X_{t_j}, Y_{t_j}))\,\mathrm{d}s$
- $f(s, X_s, Y_s) - f(t_j(s), X_{t_j(s)}, Y_{t_j(s)}) = \displaystyle\int_{t_j(s)}^{s} \mathrm{d}F_s$
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$$
\begin{aligned}
\epsilon &= \int_0^t (f(s, X_s, Y_s) - f(t_j(s), X_{t_j(s)}, Y_{t_j(s)}))\,\mathrm{d}s = \int_0^t \int_{t_j(s)}^{s} \mathrm{d}F_\tau\,\mathrm{d}s \\
&= \int_0^t \int_\tau^{t_j(\tau)+\Delta t} \mathrm{d}s\,\mathrm{d}F_\tau \ \text{(using Fubini)} \\
&= \int_0^t (t_j(\tau) + \Delta t - \tau)\,\mathrm{d}F_\tau \leq C\Delta t \ \text{(Chain rule, Itô formula, etc.)}
\end{aligned}
$$

- Hence order 1

Euler for RODEs - Itô diffusion

- RODE augmented with SDE

$$\begin{cases} \mathrm{d}X_t = f(t, X_t, Y_t)\,\mathrm{d}t \\ \mathrm{d}Y_t = \mu(t, Y_t)\,\mathrm{d}t + \sigma(t, Y_t)\,\mathrm{d}W_t \end{cases}$$

- Euler-Maruyama reduces to Euler for RODE part
- Euler-Maruyama for additive noise $\sigma = \sigma(t)$ is order 1
- Milstein method is order 1
- Milstein method reduces to Euler for RODE part
- In any case, Euler is order 1
- See Wang, Cao, Han, & P. Kloeden (2021)

Global error

- Actually work with the global error

$$X_{t_j} - X_{t_j}^N = X_0 - X_0^N + \int_0^{t_j} \left(f(s, X_s, Y_s) - f(t_i, X_{t_i}, Y_{t_i})\right) \mathrm{d}s$$

- The problematic term is the last one.

Global error

- Actually work with the global error

$$
\begin{aligned}
X_{t_j} - X_{t_j}^N &= X_0 - X_0^N + \int_0^{t_j} \left(f(s, X_s, Y_s) - f(t_i, X_{t_i}, Y_{t_i})\right) \mathrm{d}s \\
&= X_0 - X_0^N + \int_0^{t_j} \left(f(s, X_s, Y_s) - f(\tau^N(s), X_{\tau^N(s)}, Y_{\tau^N(s)})\right) \mathrm{d}s
\end{aligned}
$$

- The problematic term is the last one.

Global error

- Actually work with the global error

$$
\begin{aligned}
X_{t_j} - X_{t_j}^N &= X_0 - X_0^N + \int_0^{t_j} (f(s, X_s, Y_s) - f(t_i, X_{t_i}, Y_{t_i}))\,\mathrm{d}s \\
&= X_0 - X_0^N + \int_0^{t_j} \left(f(s, X_s, Y_s) - f(\tau^N(s), X_{\tau^N(s)}, Y_{\tau^N(s)})\right)\mathrm{d}s \\
&= X_0 - X_0^N + \int_0^{t_j} \left(f(s, X_s, Y_s) - f(s, X_{\tau^N(s)}, Y_s)\right)\mathrm{d}s \\
&\quad + \int_0^{t_j} \left(f(s, X_{\tau^N(s)}, Y_s) - f(s, X_{\tau^N(s)}^N, Y_s)\right)\,\mathrm{d}s \\
&\quad + \int_0^{t_j} \left(f(s, X_{\tau^N(s)}^N, Y_s) - f(\tau^N(s), X_{\tau^N(s)}^N, Y_{\tau^N(s)})\right)\,\mathrm{d}s,
\end{aligned}
$$

- The problematic term is the last one.

Euler for RODEs - Itô process noise

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- $\frac{\mathrm{d}X_t}{\mathrm{d}t} = f(t, X_t, Y_t)$, where $\mathrm{d}Y_t = A_t\,\mathrm{d}t + B_t\,\mathrm{d}W_t$

Euler for RODEs - Itô process noise

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- $\epsilon = \displaystyle\int_0^t (f(s, X_s, Y_s) - f(t_j(s), X_{t_j(s)}, Y_{t_j(s)}))\,\mathrm{d}s$ (global error)

Euler for RODEs - Itô process noise

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- $f(s, X_s, Y_s) - f(t_j(s), X_{t_j(s)}, Y_{t_j(s)}) = \int_{t_j(s)}^s \tilde{A}_\tau\,\mathrm{d}\tau + \int_{t_j}^s \tilde{B}_\tau\,\mathrm{d}W_\tau$

Euler for RODEs - Itô process noise

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Euler for RODEs - Itô process noise

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- $\epsilon = \displaystyle\int_0^t (f(s, X_s, Y_s) - f(t_j(s), X_{t_j(s)}, Y_{t_j(s)}))\,\mathrm{d}s$

$$= \int_0^t \int_{t_j(s)}^s \tilde{A}_\tau\,\mathrm{d}\tau + \int_{t_j}^\tau \tilde{B}_\tau\,\mathrm{d}W_\tau\,\mathrm{d}s$$

Euler for RODEs - Itô process noise

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- $\epsilon = \int_0^t (f(s, X_s, Y_s) - f(t_j(s), X_{t_j(s)}, Y_{t_j(s)}))\,\mathrm{d}s$

$$
\begin{aligned}
&= \int_0^t \int_{t_j(s)}^s \tilde{A}_\tau\,\mathrm{d}\tau + \int_{t_j}^\tau \tilde{B}_\tau\,\mathrm{d}W_\tau\,\mathrm{d}s \\
&= \int_0^t \int_\tau^{t_j(\tau)+\Delta t} \tilde{A}_\tau\,\mathrm{d}s\,\mathrm{d}\tau + \int_0^t \int_\tau^{t_j(\tau)+\Delta t} \tilde{B}_\tau\,\mathrm{d}s\,\mathrm{d}W_\tau
\end{aligned}
$$

Euler for RODEs - Itô process noise

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- $\frac{\mathrm{d}X_t}{\mathrm{d}t} = f(t, X_t, Y_t)$, where $\mathrm{d}Y_t = A_t \,\mathrm{d}t + B_t \,\mathrm{d}W_t$
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- $\epsilon = \int_0^t (f(s, X_s, Y_s) - f(t_j(s), X_{t_j(s)}, Y_{t_j(s)}))\,\mathrm{d}s$

$$= \int_0^t \int_{t_j(s)}^s \tilde{A}_\tau \,\mathrm{d}\tau + \int_{t_j}^\tau \tilde{B}_\tau \,\mathrm{d}W_\tau \,\mathrm{d}s$$

$$= \int_0^t \int_\tau^{t_j(\tau)+\Delta t} \tilde{A}_\tau \,\mathrm{d}s \,\mathrm{d}\tau + \int_0^t \int_\tau^{t_j(\tau)+\Delta t} \tilde{B}_\tau \,\mathrm{d}s \,\mathrm{d}W_\tau$$

- $\Rightarrow \mathbb{E}[|\epsilon|] \leq \left(\int_0^t \mathbb{E}[|A_\tau|] \,\mathrm{d}\tau + \left(\int_0^t \mathbb{E}[B_\tau|^2] \,\mathrm{d}\tau \right)^{1/2} \right) \Delta t.$

Euler for RODEs - Noise with bounded variation

- Finite variation process:

$$
\begin{aligned}
& f(s, X^N_{\tau^N(s)}, Y_s) - f(\tau^N(s), X^N_{\tau^N(s)}, Y_{\tau^N(s)}) \\
& = \int_{\tau^N(s)}^s D_\xi f(\xi, X^N_{\tau^N(s)}, Y_{\xi-})\,\mathrm{d}\xi \\
& \quad + \int_{\tau^N(s)^+}^s D_y f(\xi, X^N_{\tau^N(s)}, Y_{\xi-})\,\mathrm{d}Y_\xi \\
& \quad + \sum_{\tau^N(s)<\xi\leq s} \bigg(f(\xi, X^N_{\tau^N(s)}, Y_\xi) - f(\xi, X^N_{\tau^N(s)}, Y_{\xi-}) \\
& \qquad\qquad - D_y f(\xi, X^N_{\tau^N(s)}, Y_{\xi-})\Delta Y_\xi \bigg).
\end{aligned}
$$

- Same idea, just use Fubini to get order 1, using $\int_0^t |\,\mathrm{d}Y_\xi| < \infty$

Euler for RODEs - Noise semi-martingale

- $\dfrac{\mathrm{d}X_t}{\mathrm{d}t} = f(t, X_t, Y_t)$
- $\{Y_t\}_t$ semi-martingale
- $Y_t = Y_0 + F_t + Z_t$ (finite variation + local martingale)
- Essentially combine the ideas
- Use Fubini to move "irregularity"to large scales
- Not smooth but integrable somehow
- Get order 1
- The "chain rule formula" (next slide) is more involved but it works...

Global error via chain rule for semimartingales

$$
\begin{aligned}
\int_0^{t_j} & \left(f(s, X^N_{\tau^N(s)}, Y_s) - f(\tau^N(s), X^N_{\tau^N(s)}, Y_{\tau^N(s)}) \right) \mathrm{d}s \\
&= \int_0^{t_j} \int_{\tau^N(s)}^s D_\xi f(\xi, X^N_{\tau^N(s)}, Y_{\xi-}) \,\mathrm{d}\xi \,\mathrm{d}s \\
&\quad + \int_0^{t_j} \int_{\tau^N(s)^+}^s D_y f(\xi, X^N_{\tau^N(s)}, Y_{\xi-}) \,\mathrm{d}Y_\xi \,\mathrm{d}s \\
&\quad + \int_0^{t_j} \sum_{\tau^N(s) < \xi \leq s} \Big(f(\xi, X^N_{\tau^N(s)}, Y_\xi) - f(\xi, X^N_{\tau^N(s)}, Y_{\xi-}) \\
&\qquad\qquad - D_y f(\xi, X^N_{\tau^N(s)}, Y_{\xi-}) \Delta Y_\xi \Big) \,\mathrm{d}s \\
&\quad + \frac{1}{2} \int_0^{t_j} \int_{\tau^N(s)}^s D_{yy} f(\xi, X^N_{\tau^N(s)}, Y_{\xi-}) \,\mathrm{d}[Y, Y]^c_\xi \,\mathrm{d}s,
\end{aligned}
$$

where $[Y, Y]^c_\xi$ is the continuous part of the quadratic variation of the process.

Main result

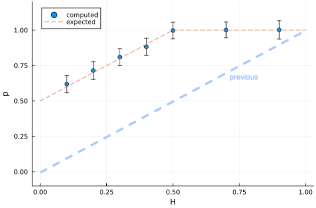
- RODE
$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = f(t, X_t, Y_t)$$
- $f : I \times \mathbb{R}^d \times \mathbb{R}^k$, with bounded derivatives $\partial_t f$, $D_x f$, $D_y f$, $D_{yy} f$
- $\mathbb{E}[\|X_0\|] < \infty$
- Semi-martingale noise $\{Y_t\}_{t\geq 0}$ in $\mathbb{R}^k$, $Y_t = Y_0 + F_t + Z_t$, with
  - $\mathbb{E}[\|Y_0\|] < \infty$
  - $\mathbb{E}\left[V(\{F_t\}_t; I)^2\right] < \infty$
  - $\mathbb{E}\left[\|Z_T\|^2\right] < \infty$
- Get strong order 1 convergence:
$$\max_{j=0,\ldots,N} \mathbb{E}\left[\left\|X_{t_j} - X_{t_j}^N\right\|\right] \leq c\Delta t_N, \qquad \forall N \in \mathbb{N},$$
- Can relax global conditions on f on bounded invariant regions
- Can relax global conditions of f for pathwise approximations

Fractional Brownian motion noise

- Only done for the linear equation

$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = -X_t + B_t^{(H)}$$

- Same idea of using Fubini
- But local error has a $(t-s)^{H-1/2}$ kernel
- This leads to an $\mathcal{O}(\Delta t + \Delta t^{H+1/2})$ error
- For $H = 1/2$, fBm is Wiener, recover order 1
- For $1/2 < H \leq 1$, it is smoother, also get order 1
- For $0 < H < 1/2$, get lower order $H + 1/2 < 1$, but still better than previously know order H.



Outlook

- General result for fBm (not just that linear equation)
- Other non-semimartingale noises (e.g. Volterra stochastic noise)
- Higher order methods for non-Itô noises
- Random PDEs
- Pathwise convergence (work with Luan Lima Freitas)

Thank you!