

Energy cascade for cross-shear length scales in free-shear three-dimensional incompressible viscous flows

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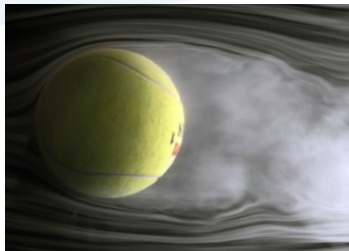


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Fluid flows displaying shear effects

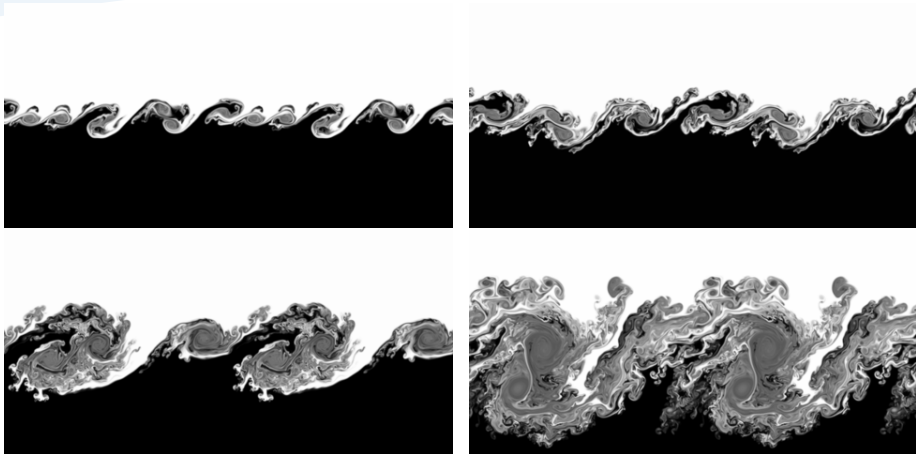


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Kelvin-Helmholtz instability

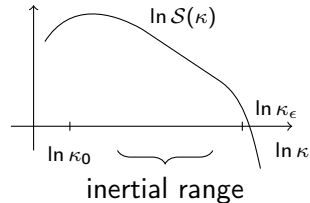
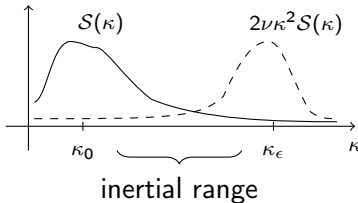




Energy cascade mechanism

In an inertial range of scales:

- ▶ Length scale $\sim 1/\kappa$
- ▶ Energy $e_\kappa \sim \mathcal{S}(\kappa)\kappa$
- ▶ Time scale $t_\kappa \sim \frac{1}{\kappa\sqrt{e_\kappa}}$
- ▶ Energy flux to smaller scales $\varphi_\kappa \sim \frac{e_\kappa}{t_\kappa}$
- ▶ In statistical equilibrium, flux $\varphi_\kappa \sim \epsilon$
- ▶ Therefore $\mathcal{S}(\kappa) \sim \epsilon^{2/3}\kappa^{-5/3}$.



Essentially **body-forced flows**.

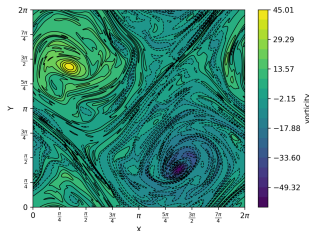
- ▶ **Foias, Manley, R., Temam (2001)**, Estimates for the energy cascade in three-dimensional turbulent flows.
- ▶ **Foias, Manley, R., Temam (2002)**, Statistical estimates for the Navier–Stokes equations and the Kraichnan theory of 2-D fully developed turbulence.
- ▶ **Dascaliuc, Grujić (2011)**, Energy cascades and flux locality in physical scales of the 3D NSE.
- ▶ **Dascaliuc, Grujić (2012)**, Dissipation anomaly and energy cascade in 3D incompressible flow.
- ▶ **Dascaliuc, Grujić (2013)**, Anomalous dissipation and energy cascade in 3D inviscid flows.
- ▶ **Jolly, Wirosoetisno (2018)**, Energy spectra and passive tracer cascades in turbulent flows.

Body-forced mixed deterministic-stochastic flows:

- ▶ Fu, Jolly, R. (2026), Analytical and numerical results for the two-dimensional Stochastic Navier–Stokes equations, *in preparation*.

Free-shear flows:

- ▶ R. (2026), Energy cascade for cross-shear length scales in free-shear three-dimensional incompressible viscous flows, *submitted*.
- ▶ R., Energy cascade in free-shear three-dimensional incompressible viscous flows, *in preparation*.



- Momentum equations:

$$\frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = \mathbf{f}$$

- Assume f is concentrated on low modes and look at the energy budget equations within different modes.

Energy cascade in shear flows?



- ▶ Look at perturbations from the mean flow $\mathbf{v} = \mathbf{U} + \mathbf{u}$.
- ▶ Assuming $\mathbf{U} = (U(z), 0, 0)$, the momentum equations for the perturbation become:

$$\frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = -U(z) \frac{\partial \mathbf{u}}{\partial x} - w U'(z) \mathbf{e}_x.$$

- ▶ For free-shear flows, assume mean flow profile $U = U(z)$ is given.
- ▶ But can't simply assume forcing is in low modes.
- ▶ Need to estimate advection and shear terms.

- ▶ Foias, Manley, R., Temam (2001)

- ▶ 3D NSE

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = \mathbf{f}, \\ \nabla \cdot \mathbf{u} = 0. \end{cases}$$

- ▶ Either periodic boundary conditions or no slip.
- ▶ Body force $\mathbf{f}$ restricted to low modes $\kappa_0 \leq \kappa \leq \kappa_f$.
- ▶ Smallest wavenumber denoted κ_0 used for dimensionality.

Energy cascade in forced flows

- ▶ Considering the component $u_{\kappa,\infty}$ of the flow with wavenumbers larger than κ , the energy budget equation reads:

$$\frac{1}{2} \frac{d}{dt} \|u_{\kappa,\infty}\|^2 + \nu \|\nabla u_{\kappa,\infty}\|^2 + (((u \cdot \nabla) u_{\kappa,\infty}, u_{\kappa,\infty})) = ((f, u_{\kappa,\infty})).$$

- ▶ Taking the ensemble average (or asymptotic time average) and assuming f is concentrated on low modes, we get:

$$\nu \langle \|\nabla u_{\kappa,\infty}\|^2 \rangle = -(((u \cdot \nabla) u_{\kappa,\infty}, u_{\kappa,\infty})).$$

- ▶ Multiplying by κ_0^3 the right hand side becomes the energy flux Π_k towards higher modes

$$\Pi_\kappa = -\kappa_0^3 (((u \cdot \nabla) u_{\kappa,\infty}, u_{\kappa,\infty})).$$

- ▶ The equality in the energy equation holds for regular solutions and we bound the energy flux by the energy dissipation rate ϵ :

$$\Pi_\kappa = \nu \kappa_0^3 \langle \|\nabla u_{\kappa,\infty}\|^2 \rangle \leq \nu \kappa_0^3 \langle \|\nabla u\|^2 \rangle = \epsilon.$$

Energy cascade in forced flows

- On the other hand, for arbitrary weak solutions,

$$\begin{aligned}
 \Pi_\kappa &\geq \nu \kappa_0^3 \langle \|\nabla \mathbf{u}_{\kappa, \infty}\|^2 \rangle \\
 &= \nu \kappa_0^3 \langle \|\nabla \mathbf{u}\|^2 \rangle - \nu \kappa_0^3 \langle \|\nabla \mathbf{u}_{0, \kappa}\|^2 \rangle \\
 &\geq \nu \kappa_0^3 \langle \|\nabla \mathbf{u}\|^2 \rangle - \nu \kappa_0^3 \kappa^2 \langle \|\mathbf{u}_{0, \kappa}\|^2 \rangle \\
 &\geq \nu \kappa_0^3 \langle \|\nabla \mathbf{u}\|^2 \rangle - \nu \kappa_0^3 \kappa^2 \langle \|\mathbf{u}\|^2 \rangle \\
 &= \nu \kappa_0^3 \langle \|\nabla \mathbf{u}\|^2 \rangle - \nu \kappa_0^3 \kappa^2 \frac{\langle \|\mathbf{u}\|^2 \rangle}{\langle \|\nabla \mathbf{u}\|^2 \rangle} \langle \|\nabla \mathbf{u}\|^2 \rangle.
 \end{aligned}$$

- In other words, for $\kappa_f^2 \leq \kappa^2 \ll \kappa_\tau^2$,

$$\Pi_\kappa \geq \left(1 - \frac{\kappa^2}{\kappa_\tau^2}\right) \epsilon \gtrsim \epsilon,$$

where $\kappa_\tau = \left(\frac{\langle \|\mathbf{u}\|^2 \rangle}{\langle \|\nabla \mathbf{u}\|^2 \rangle}\right)^{1/2}$ is the Taylor microscale wavenumber.

- For regular solutions, get a more precise description of the cascade in the inertial range $\kappa_f^2 \leq \kappa^2 \ll \kappa_T^2$:

$$\Pi_\kappa \approx \epsilon.$$

- For weak solutions, can remove possible loss of energy “to infinity”

$$\Pi_\kappa^* = \Pi_\kappa - \lim_{\kappa' \rightarrow \infty} \Pi_{\kappa'}$$

and obtain also

$$\Pi_\kappa^* \approx \epsilon.$$

Forcing term concentrated on low modes

- If the forcing is concentrated on but not restricted to low modes, get

$$\Pi_\kappa \geq \nu \kappa_0^3 \langle \|\nabla \mathbf{u}_{\kappa,\infty}\|^2 \rangle - \kappa_0^3 \langle ((\mathbf{f}_{\kappa,\infty}, \mathbf{u}_{\kappa,\infty})) \rangle$$

with, still,

$$\begin{aligned} \kappa_0^3 \langle ((\mathbf{f}_{\kappa,\infty}, \mathbf{u}_{\kappa,\infty})) \rangle &\leq \frac{\kappa_0^{3/2}}{\nu^{1/2}} \langle \|A^{-1/2} \mathbf{f}_{\kappa,\infty}\|^2 \rangle^{1/2} \left(\nu \kappa_0^3 \langle \|A^{1/2} \mathbf{u}_{\kappa,\infty}\|^2 \rangle \right)^{1/2} \\ &\leq \left(\frac{\kappa_0^3}{\nu} \frac{\|A^{-1/2} \mathbf{f}_{\kappa,\infty}\|}{\|A^{-1/2} \mathbf{f}\|} \|A^{-1/2} \mathbf{f}\| \right)^{1/2} \left(\nu \kappa_0^3 \langle \|A^{1/2} \mathbf{u}_{\kappa,\infty}\|^2 \rangle \right)^{1/2} \\ &\leq \dots \ll \epsilon \end{aligned}$$

provided $\|A^{-1/2} \mathbf{f}_{\kappa,\infty}\| \ll \|A^{-1/2} \mathbf{f}\|$ decays sufficiently fast to have an inertial range below κ_σ .

- ▶ On channel flows, the pressure drop term is not concentrated on low modes

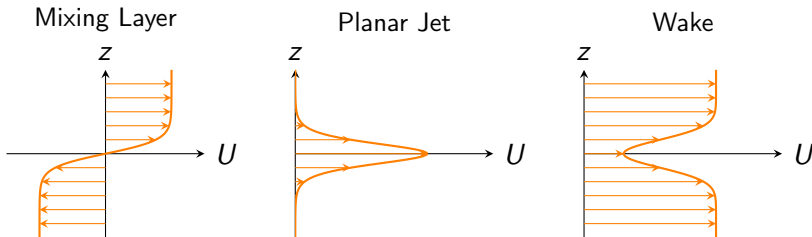
$$\frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = \frac{P}{L_x} \mathbf{e}_x.$$

- ▶ and the effect of the pressure drop decays slowly in Fourier space $\sim 1/\kappa$.
- ▶ Even on free-shear flows, the decay is problematic.
- ▶ How can we handle the extra terms in the energy budget equation?

$$\left\{ \begin{array}{l} \frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = -U(z) \frac{\partial \mathbf{u}}{\partial x} - w U'(z) \mathbf{e}_x, \\ \nabla \cdot \mathbf{u} = 0, \\ u(x + L_x \mathbf{e}_x) = u(x), \quad p(x + L_x \mathbf{e}_x) = p(x), \\ u(x + L_y \mathbf{e}_y) = u(x), \quad p(x + L_y \mathbf{e}_y) = p(x), \\ \frac{\partial u}{\partial z}(x, y, \pm h/2) = 0, \quad \frac{\partial v}{\partial z}(x, y, \pm h/2) = 0, \quad w(x, y, \pm h/2) = 0, \\ \int_0^{L_x} \int_0^{L_y} u(x, y, z) \, dx \, dy = 0, \quad \int_0^{L_x} \int_0^{L_y} v(x, y, z) \, dx \, dy = 0. \end{array} \right.$$



► Examples:



► Assume mean flow profile $U = U(z)$ is given and satisfies:

$$U(z) \in C^1([-h/2, h/2]),$$
$$U'(\pm h/2) = 0.$$

- ▶ $S = \|U'\|_{L^\infty}$ measures the **shear rate**.
- ▶ $1/S$ is regarded as the **mean shear time**.
- ▶ $\ell_S = (\nu/S)^{1/2}$ is the **viscous shear length scale**, measuring the size at which the viscous diffusion time equals the shear timescale, marking the lower cutoff below which viscosity smears out shear-driven velocity variations faster than the mean shear can build them up.
- ▶ $\ell_C = (\epsilon/S^3)^{1/2}$ is the **Corrsin length scale**, measuring the eddy size at which the turbulent turnover time equals the shear timescale, so that eddies larger than ℓ_C are stretched and rendered anisotropic by the mean shear while smaller eddies turn over too fast to feel it and stay effectively isotropic.

- In this geometry, $Aw_{j,l,k,\iota} = \lambda_{j,l,k} w_{j,l,k,\iota}$, for indices in

$$\mathcal{I} = \{(j, l, k, \iota); j, l \in \mathbb{Z}, k \in \mathbb{N}_0, \iota \in \{1, 2\}, j^2 + l^2 \neq 0\},$$

with eigenvalues

$$\lambda_{j,l,k} = \frac{4\pi^2 j^2}{L_x^2} + \frac{4\pi^2 l^2}{L_y^2} + \frac{k^2 \pi^2}{h^2}, \quad j, l \in \mathbb{Z}, k \in \mathbb{N}_0, j^2 + l^2 \neq 0.$$

and eigenfunctions

$$w_{j,l,k,\iota}(x, y, z) = \begin{pmatrix} u_{j,l}^h(x, y) u_{j,l,k,\iota}^v(z) \\ v_{j,l}^h(x, y) v_{j,l,k,\iota}^v(z) \\ w_{j,l}^h(x, y) w_{j,l,k,\iota}^v(z) \end{pmatrix},$$

- We look at the horizontal wavenumbers

$$\bar{\kappa}_{j,l} = \sqrt{\frac{4\pi^2 j^2}{L_x^2} + \frac{4\pi^2 l^2}{L_y^2}}, \quad j, l \in \mathbb{Z}, j^2 + l^2 \neq 0.$$

- And decompose the flow with respect to the horizontal wavenumber modes:

$$\mathbf{u}_{\bar{\kappa},\infty} = \sum_{\bar{\kappa}_{j,l} \geq \bar{\kappa}} \sum_{k \in \mathbb{N}_0} \sum_{\ell=1}^2 ((\mathbf{u}, \mathbf{w}_{j,l,k,\ell})) \mathbf{w}_{j,l,k,\ell}.$$

- The energy budget equations read

$$\begin{aligned} \nu \kappa_0^3 \langle \|\nabla \mathbf{u}_{\bar{\kappa},\infty}\|^2 \rangle + \kappa_0^3 \langle (((\mathbf{u} \cdot \nabla) \mathbf{u}_{0,\bar{\kappa}}, \mathbf{u}_{\bar{\kappa},\infty})) \rangle \\ \leq -\kappa_0^3 \langle ((U(z) \partial_x \mathbf{u}, \mathbf{u}_{\bar{\kappa},\infty})) \rangle - \kappa_0^3 \langle ((w U'(z) \mathbf{e}_x, \mathbf{u}_{\bar{\kappa},\infty})) \rangle. \end{aligned}$$

- ▶ The advection term is orthogonal to the horizontal modes.
- ▶ Writing $u = u_{0,\bar{k}} + u_{\bar{k},\infty}$, have

$$\begin{aligned} & \langle ((U(z)\partial_x u_{0,\bar{k}}, u_{\bar{k},\infty})) \rangle \\ &= \int_{-h/2}^{h/2} U(z) \int_0^{L_y} \int_0^{L_x} \partial_x u_{0,\bar{k}}(x, y, z) \cdot u_{\bar{k},\infty}(x, y, z) \, dx \, dy \, dz = 0, \end{aligned}$$

and

$$\begin{aligned} & \langle ((U(z)\partial_x u_{\bar{k},\infty}, u_{\bar{k},\infty})) \rangle \\ &= \int_{-h/2}^{h/2} U(z) \int_0^{L_y} \int_0^{L_x} \partial_x u_{\bar{k},\infty}(x, y, z) \cdot u_{\bar{k},\infty}(x, y, z) \, dx \, dy \, dz \\ &= \int_{-h/2}^{h/2} U(z) \int_0^{L_y} \int_0^{L_x} \frac{1}{2} \partial_x (u_{\bar{k},\infty}(x, y, z))^2 \, dx \, dy \, dz = 0. \end{aligned}$$

Estimating the production term

$$\begin{aligned}
 \mathcal{P}_{\bar{\kappa},\infty} &= -\kappa_0^3 \langle ((wU'(z)e_x, u_{\bar{\kappa},\infty})) \rangle \\
 &= -\kappa_0^3 \langle ((wU'(z)e_x, u_{\bar{\kappa},\infty}e_x)) \rangle \\
 &= -\kappa_0^3 \langle ((w_{\bar{\kappa},\infty}U'(z)e_x, u_{\bar{\kappa},\infty}e_x)) \rangle \\
 &\leq \kappa_0^3 \|U'\|_{L^\infty} \langle \|w_{\bar{\kappa},\infty}\| \|u_{\bar{\kappa},\infty}\| \rangle \\
 &\leq \frac{\kappa_0^3}{2} \|U'\|_{L^\infty} \langle \|u_{\bar{\kappa},\infty}\|^2 \rangle \\
 &\leq \frac{1}{2} \frac{\langle \|u_{\bar{\kappa},\infty}\|^2 \rangle}{\langle \|\nabla_{x,y} u_{\bar{\kappa},\infty}\|^2 \rangle} \frac{\|U'\|_{L^\infty}}{\nu} \nu \kappa_0^3 \langle \|\nabla_{x,y} u_{\bar{\kappa},\infty}\|^2 \rangle \\
 &\leq \frac{\kappa_S^2}{2\mathcal{K}_{\bar{\kappa},\infty}^2} \epsilon_{\bar{\kappa},\infty}^{x,y}.
 \end{aligned}$$

where $\kappa_S = 1/\ell_S = (S/\nu)^{1/2}$ is the viscous shear wavenumber, and $\mathcal{K}_{\bar{\kappa},\infty} = (\langle \|\nabla_{x,y} u_{\bar{\kappa},\infty}\|^2 \rangle / \langle \|u_{\bar{\kappa},\infty}\|^2 \rangle)^{1/2}$ is a scale-dependent Taylor-like microscale wavenumber.

Energy cascade

- Back to the energy budget equations

$$\begin{aligned}
 \Pi_{\bar{\kappa}} &\geq \epsilon_{\bar{\kappa},\infty} - \mathcal{P}_{\bar{\kappa},\infty} \geq \epsilon_{\bar{\kappa},\infty} - \frac{\kappa_S^2}{2\mathcal{K}_{\bar{\kappa},\infty}^2} \epsilon_{\bar{\kappa},\infty}^{x,y} = \epsilon_{\bar{\kappa},\infty}^z + \epsilon_{\bar{\kappa},\infty}^{x,y} - \frac{\kappa_S^2}{2\mathcal{K}_{\bar{\kappa},\infty}^2} \epsilon_{\bar{\kappa},\infty}^{x,y} \\
 &= \epsilon_{\bar{\kappa},\infty}^z + \left(1 - \frac{\kappa_S^2}{2\mathcal{K}_{\bar{\kappa},\infty}^2}\right) \epsilon_{\bar{\kappa},\infty}^{x,y} \\
 &= \left(1 - \frac{\kappa_S^2}{2\mathcal{K}_{\bar{\kappa},\infty}^2}\right) \frac{\epsilon_{\bar{\kappa},\infty}}{\epsilon} \epsilon = \left(1 - \frac{\kappa_S^2}{2\mathcal{K}_{\bar{\kappa},\infty}^2}\right) \frac{\epsilon - \epsilon_{0,\bar{\kappa}}}{\epsilon} \epsilon \\
 &= \left(1 - \frac{\kappa_S^2}{2\mathcal{K}_{\bar{\kappa},\infty}^2}\right) \left(1 - \frac{\epsilon_{0,\bar{\kappa}}}{\epsilon}\right) \epsilon,
 \end{aligned}$$

- Thus $\Pi_{\bar{\kappa}} \gtrsim \epsilon$ for $\bar{\kappa}$ satisfying

$$\bar{\kappa}_S^2 \ll \mathcal{K}_{\bar{\kappa},\infty}^2 \quad \text{and} \quad \epsilon_{0,\bar{\kappa}} \ll \epsilon.$$

- Similarly, $\Pi_{\bar{\kappa}}^* \approx \epsilon$.

Energy cascade conditions

- The conditions are not as explicit:

$$\bar{\kappa}_S^2 \ll \mathcal{K}_{\bar{\kappa},\infty}^2 \quad \text{and} \quad \epsilon_{0,\bar{\kappa}} \ll \epsilon.$$

- But if we assume the $\kappa^{-5/3}$ spectrum holds within the inertial range, then $\mathcal{K}_{\bar{\kappa},\infty}^2 \sim \kappa_\eta^{4/3} \bar{\kappa}^{2/3}$, and the first condition becomes

$$\bar{\kappa} \gg \frac{\kappa_S^3}{\kappa_\eta^2} = \frac{(S/\nu)^{3/2}}{(\epsilon/\nu^3)^{1/2}} = \frac{S^{3/2}}{\epsilon^{1/2}} = \kappa_C,$$

where $\kappa_C = 1/\ell_C$ is the Corrsin wavenumber.

- The second condition is satisfied for $\kappa \ll \kappa_\eta$.
- Thus, the energy cascade conditions translates to precisely the classical inertial range conditions for turbulent shear flows:

$$\kappa_C \ll \bar{\kappa} \ll \kappa_\eta.$$

Obrigado!
Grazie!

